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ESTIMATING BOUNDS ON THE SOLUTIONS OF TWO KINDS OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

The aim of the present paper is to estimate bounds on the solutions of two kinds of nonlinear integro-differential equations, the first one is ordinary and the other is partial.

1. Introduction

The integral inequalities involving functions of one and two independent variables, which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of differential equations [2, 4]. During the past few years, many such new inequalities have been discovered [1, 3, 5], this paper is to obtain estimates of upper bound of two kinds of integro-differential equations, the first one is nonlinear ordinary integro-differential equation and the other is hyperbolic partial integro-differential equation.

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We need the following theorems:

Theorem A [6]. *Let* $u, f, g,$ *and* h *be nonnegative continuous functions defined on* R_+ *and* c *be a nonnegative constant. If*

$$u^2(t) \leq c^2 + 2\int_0^t [f(s)u(s)(u(s) + \int_0^s g(\alpha)u(\alpha)d\alpha) + h(s)u(s)]ds, \tag{1.1}$$

for $t \in R_+$, *then*

$$u(t) \leq p(t)\left[1 + \int_0^t [f(s)\exp\left(\int_0^s [f(\alpha) + g(\alpha)]d\alpha\right)]ds\right], \tag{1.2}$$

for $t \in R_+$, *where* $p(t) = c + \int_0^t h(s)ds.$

Theorem B [6]. *Assume that* $u(x, y), a(x, y),$ *and* $b(x, y)$ *are nonnegative continuous functions defined for* $x, y \in R_+$. *Let* $g(u)$ *be continuously differentiable function defined for* $u \geq 0, g(u) > 0$ *for* $u > 0,$ *and* $g'(u) \geq 0$ *for* $u \geq 0$ *and* $g(u)$ *be subadditive on* R_+. *If*

$$u(x, y) \leq a(x, y) + \int_0^x \int_0^y b(s, t)\left(u(s, t) + \int_0^s \int_0^t b(s_1, t_1)g(u(s_1, t_1))ds_1 dt_1\right)dsdt, \tag{1.3}$$

then, for $0 \leq x \leq x_1, 0 \leq y \leq y_1,$

$$u(x, y) \leq a(x, y) + A(x, y) + \int_0^x \int_0^y b(s, t)\left\{H^{-1}\left[H\left(A(s, t) + \int_0^s \int_0^t p(s_1, t_1)\, ds_1 dt_1\right)\right]\right\}dsdt, \tag{1.4}$$

where

$$A(x, y) = \int_0^x \int_0^y c(s_1, t_1)\left(a(s_1, t_1) + \int_0^{s_1} \int_0^{t_1} b(s_2, t_2)g(a(s_2, t_2))ds_2 dt_2\right)ds_1 dt_1,$$

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(h(s))}, \ r > 0, \ r_0 > 0,$$

and

$$H(r) = \int_{r_0}^{r} \frac{ds}{s + g(s)}, \ r > 0, \ r_0 > 0,$$

H^{-1} *is the inverse function of* H, *and* $x_1, \ y_1$ *are chosen so that*

$$H(A(x, \ y)) + \int_0^x \int_0^y p(s_1, \ t_1) ds_1 dt_1 \in \operatorname{dom}(H^{-1}),$$

for all x, y *lying in the subintervals* $0 \le x \le x_1, \ 0 \le y \le y_1$ *of* R_+.

2. Main Results

Theorem 1. *Consider the nonlinear integro-differential equation*

$$x'(t) - F\left(t, \ x(t), \ \int_0^t K(t, \ s, \ x(s)) ds\right) = h(t), \ x(0) = x_0, \tag{2.1}$$

where $h : R_+ \to R, \ K : R_+^2 \times R \to R, \ F : R_+ \times R^2 \to R$ *are continuous functions. If*

$$|K(t, \ s, \ x(s))| \le f(t) g(s) |x(s)|, \tag{2.2}$$

$$|F(t, \ x(t), \ v(t))| \le f(t)|x(s)| + |v(s)|, \tag{2.3}$$

where f *and* g *are real-valued nonnegative continuous functions defined on* R_+, *then the solution of problem* (2.1) *is bounded.*

Proof. Multiplying both sides of Equation (2.1) by $x(t)$, we get

$$x(t)x'(t) - x(t)F\left(t, \ x(t), \ \int_0^t K(t, \ s, \ x(s)) ds\right) = x(t)h(t). \tag{2.4}$$

Put $t = s$ and integrate from 0 to t, we obtain

$$\int_0^t x(s)x'(s)ds - \int_0^t x(s)F\left(s,\ x(s),\ \int_0^s K(s,\ \tau,\ x(\tau))d\tau\right)ds = \int_0^t x(s)h(s)ds. \tag{2.5}$$

Using the initial condition

$$x^2 = x_0^2 + 2\int_0^t [x(s)F(s,\ x(s),\ \int_0^s K(s,\ \tau,\ x(\tau))d\tau) + x(s)h(s)]ds. \tag{2.6}$$

We obtain

$$|x|^2 = |x_0|^2 + 2\int_0^t [|x(s)|\left|F\left(s,\ x(s),\ \int_0^s K(s,\ \tau,\ x(\tau))d\tau\right)\right| + |x(s)||h(s)|]ds. \tag{2.7}$$

Using condition (2.3), we get

$$|x|^2 = |x_0|^2 + 2\int_0^t [|x(s)||f(s)||x(s)| + \left|\int_0^s K(s,\ \tau,\ x(\tau))d\tau\right| + |x(s)||h(s)|]ds. \tag{2.8}$$

Using condition (2.2), we get

$$|x|^2 = |x_0|^2 + 2\int_0^t |x(s)||f(s)|[|x(s)| + \left|\int_0^s g(\tau)x(\tau)d\tau\right| + |x(s)||h(s)|]ds. \tag{2.9}$$

Applying Theorem (A), we have

$$x(t) \le p_1(t)[1 + \int_0^t [f(s)\exp(\int_0^s [f(\tau) + g(\tau)]d\tau)]ds], \tag{2.10}$$

where $p_1(t) = |x_0| + \int_0^t |h(s)|ds,\ t \in R_+$. Inequality (2.10) gives the bound on the solution $x(t)$ of Equation (2.1) in terms of known functions.

Theorem 2. *Consider the boundary value problem of nonlinear hyperbolic partial integro-differential equation*:

$$\frac{\partial^2 u}{\partial x \partial y} = f(x,\ y,\ u(x,\ y),\ \int_0^x \int_0^y k(x,\ y,\ s,\ t,\ u(s,\ t))dsdt,$$

$$u(x,\ 0) = \sigma(x), \quad u(0,\ y) = \tau(y),\ \textit{with}\ u(0,\ 0) = 0, \tag{2.11}$$

where $\sigma, \tau : R_+ \to R$, $k : R_+^4 \times R \to R$, $f : R_+^2 \times R \times R \to R$ *are continuous functions, if*

$$|\sigma(x) + \tau(y)| \leq a(x,\ y), \tag{2.12}$$

$$|k(x,\ y,\ s,\ t,\ u)| \leq b(s,\ t)g(|u|), \tag{2.13}$$

$$|f(x,\ y,\ u,\ v)| \leq b(x,\ y)[\,|u| + |v|\,], \tag{2.14}$$

where $a(x,\ y)$, $b(x,\ y)$, *and* $g(r)$ *are as defined in* R_+ *then the solution of problem* (2.1) *is bounded.*

Proof. Integrating both sides of (2.11) from 0 to y, we get

$$\int_0^y \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)dt = \int_0^y f(x,\ t,\ u(x,\ t) + \int_0^x \int_0^t k(x,\ t,\ s,\ t_1,\ u(s,\ t_1))dsdt_1)dt. \tag{2.15}$$

Then

$$\frac{\partial u(x,\ y)}{\partial x} = \frac{\partial u(x,\ 0)}{\partial x} + \int_0^y f(x,\ t,\ u(x,\ t),\ \int_0^x \int_0^t k(x,\ t,\ s,\ t_1,\ u(s,\ t_1))dsdt_1)dt. \tag{2.16}$$

Integrating both sides of (2.16) from 0 to x with respect to s and using the boundary conditions, we get

$$u(x,\ y) = \sigma(x) + \tau(y)$$

$$+ \int_0^x \int_0^y f(s,\ t,\ u(s,\ t),\ \int_0^s \int_0^t k(s,\ t,\ s_1,\ t_1,\ u(s_1,\ t_1))ds_1 dt_1)dsdt. \tag{2.17}$$

Then

$$|u(x,\ y)| \leq |\sigma(x) + \tau(y)| + \int_0^x \int_0^y |f(s,\, t,\, u(s,\, t),\, \int_0^s \int_0^t k(s,\, t,\, s_1,\, t_1,\, u(s_1,\, t_1)) ds_1 dt_1) dsdt|. \tag{2.18}$$

Using conditions (2.12) and (2.14)

$$|u(x,\ y)| \leq a(x,\ y) + \int_0^x \int_0^y b(s,\, t)[|u(s,\, t)| + \left| \int_0^s \int_0^t k(s,\, t,\, s_1,\, t_1,\, u(s_1,\, t_1)) ds_1 dt_1] dsdt \right|. \tag{2.19}$$

Using condition (2.13), we get

$$|u(x,\ y)| \leq a(x,\ y) + \int_0^x \int_0^y b(s,\, t)[|u(s,\, t)| + \int_0^s \int_0^t b(s_1,\, t_1) g(|u(s_1,\, t_1)|) ds_1 dt_1] dsdt. \tag{2.20}$$

Applying Theorem (B), we get

$$|u(x,\ y)| \leq a(x,\ y) + A(x,\ y) + \int_0^x \int_0^y b(s,\, t)\{H^{-1}[H\left(A(s,\, t) + \int_0^s \int_0^t p(s_1,\, t_1) ds_1 dt_1\right)]\} dsdt, \tag{2.21}$$

where $A(x,\ y)$, H, H^{-1} defined as in Theorem B, i.e., the solution of problem (2.1) is bounded.

References

[1] H. Al-Jeaid, Boundedness and uniqueness of Volterra partial-linear inequalities in two independent variables, Far East J. Appl. Math. 32(3) (2008), 409-414.

[2] E. Backenback and R. Bellman, Inequalities, Springer-Verlag, Berlin, 1961.

[3] D. Bainov and P. Simelnov, Integral Inequalities and Applications, Kluwer Academic Publishers, 1992.

[4] H. EL-Owaidy and H. Al-Jeaid, Asymptotic behavior of some integro-differential equations involving higher order derivatives in two independent variables, J. Inst. Math. Comp. Sci. (Math. Ser.) 19(1) (2006), 39-44.

[5] W. M. Fan and N. L. Wei, On some new integral inequalities and their applications, App. Math. Appl. 148 (2004), 381-392.

[6] B. Pachpatte, Inequalities for Differential and Integral Equations, Academic Press, New York, 1998.

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